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A NOTE ON A FLAT TOROIDAL CRACK IN AN ELASTIC ISOTROPIC BODY

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Abstract—Betti's reciprocal theorem is used to derive integral equations for problems involving an axisymmetric flat toroidal crack. Numerical estimates for stress intensity factors in fracture mechanics are improved by use of the M-integral conservation law. Stress intensity factors for torsion and uniform tension are given.

1. INTRODUCTION

For elastic problems involving an axisymmetric crack, a powerful method of approach is the use of integral transforms, such as the Hankel transform, which usually converts the boundary value problem into the solution of integral equations[1]. Although the method of the solution involves elaborate mathematical manipulation, for some simple problems the final expressions for physically meaningful quantities are elementary. The purpose of this paper is to illustrate an alternate approach which was used by Shield[2] to find surface values for indentation problems. In this method Betti's reciprocal theorem leads to integral equations for physically meaningful quantities. Although we need auxiliary solutions, the mathematical manipulations are relatively elementary.

In engineering practice a curved crack such as a banana-shaped crack is frequently encountered. The stress intensity factors at the midpoints of the crack can be estimated by the stress intensity factors of a flat toroidal crack.

In Section 2 we give Betti's reciprocal theorem and the conservation laws for future use. The torsion of an external crack covering the outside of a circle is considered in Section 3. By use of Betti's reciprocal theorem the stress intensity factor is found, in agreement with the existing solution. We next consider a flat toroidal crack and for this problem a Fredholm integral equation of the second kind is derived. The M-integral is used to improve the accuracy of the numerical solution for the stress intensity factors. In Section 4 we treat the application of pressure on the surfaces of a flat toroidal crack. Numerical results are obtained for uniform pressure and compared with previous results.

2. BASIC FORMULAE

Consider two displacement fields u_i, u'_i associated with equilibrium states of the body. It is assumed that the fields are smooth enough that Betti's reciprocal theorem holds, which for zero body force gives

$$\int_S T_i u'_i \, dS = \int_S T'_i u_i \, dS, \tag{2.1}$$

where a repeated index implies summation over the range 1–3, S is the boundary surface of the body and T_i, T'_i are the surface tractions associated with u_i, u'_i, respectively. For infinite bodies we suppose that u_i are $O(1/\rho)$ and the stress components τ_{ij} are $O(1/\rho^2)$ as $\rho \to \infty$ where $\rho = (x_i x_i)^{1/2}$.

Conservation laws for homogeneous elastic bodies were derived by Eshelby[3] and additional laws were established by Günther[4] (see also [5]). Let A be the boundary of a regular subregion of the body with the outward normal n_i then

$$\begin{aligned} J_i &= \int_A (W n_i - T_j u_{j,i}) \, dS = 0, \\ M &= \int_A (W x_i n_i - T_j u_{j,i} x_i - \tfrac{1}{2} T_i u_i) \, dS = 0, \end{aligned} \tag{2.2}$$

where W denotes the strain-energy density. For a crack under combined loading of plane strain and antiplane shear we take the crack face to be in the x_1-direction. Then the contribution to J_1 per unit length parallel to the crack from a vanishingly small path which begins on one crack face, surrounds the crack tip, and terminates on the opposite face is $\pm J$ where

$$J = \frac{\pi}{2\mu}[(1-\nu)(k_1^2 + k_2^2) + k_3^2] \tag{2.3}$$

is Rice's integral of fracture mechanics [6] and the algebraic sign is determined from the details of the problem. In (2.3) k_i denote the stress intensity factors (see (3.9) below for their definition) and μ, ν are the shear modulus and Poisson's ratio, respectively.

3. AXISYMMETRIC TORSION OF A FLAT TOROIDAL CRACK

We first consider an external crack covering the outside of a circle of radius a in an infinite elastic body. The crack surface is subjected to an axisymmetric distribution of tangential traction. With a cylindrical coordinate system (r, θ, z), a solution exists for which u_r and u_z vanish and u_θ is independent of θ. The boundary conditions are

$$\begin{aligned} \tau_{z\theta} &= -\tau(r) \quad \text{for } r > a,\ z = 0, \\ u_\theta &= 0 \qquad\quad \text{for } 0 \le r \le a,\ z = 0, \end{aligned} \tag{3.1}$$

where $\tau(r)$ is a known function. The twisting moment produced by the traction $\tau(r)$ is assumed to be finite. Because of the symmetry of the geometry and the boundary conditions it is sufficient to consider the lower half space $z \le 0$. For the application of Betti's reciprocal theorem, consider the Reissner–Sagoci problem

$$\begin{aligned} u'_\theta &= r \quad \text{for } 0 \le r \le t,\quad z = 0, \\ \tau'_{z\theta} &= 0 \quad \text{for } r > t,\ z = 0, \end{aligned} \tag{3.2}$$

which results in surface values [7]

$$\begin{aligned} \tau'_{z\theta} &= \frac{4\mu r}{\pi(t^2 - r^2)^{1/2}} \quad \text{for } 0 \le r < t,\ z = 0, \\ u'_\theta &= \frac{2}{\pi}\left[r \sin^{-1}\left(\frac{t}{r}\right) - \frac{t}{r}(r^2 - t^2)^{1/2}\right] \quad \text{for } r > t,\ z = 0. \end{aligned} \tag{3.3}$$

We apply the reciprocal theorem (2.1) to the fields in $z \le 0$ satisfying the boundary conditions (3.1)–(3.3). Then we have for $0 \le t < a$

$$\begin{aligned} &\int_0^t \tau_{z\theta} r^2\,\mathrm{d}r + \frac{2}{\pi}\int_t^a \tau_{z\theta}\left[r \sin^{-1}\left(\frac{t}{r}\right) - \frac{t}{r}(r^2 - t^2)^{1/2}\right] r\,\mathrm{d}r \\ &\qquad - \frac{2}{\pi}\int_a^\infty \tau(r)\left[r \sin^{-1}\left(\frac{t}{r}\right) - \frac{t}{r}(r^2 - t^2)^{1/2}\right] r\,\mathrm{d}r = 0 \end{aligned} \tag{3.4}$$

and for $t > a$

$$\int_0^a \tau_{z\theta} r^2\,\mathrm{d}r - \int_a^t \tau(r) r^2\,\mathrm{d}r - \frac{2}{\pi}\int_t^\infty \tau(r)\left[r \sin^{-1}\left(\frac{t}{r}\right) - \frac{t}{r}(r^2 - t^2)^{1/2}\right] r\,\mathrm{d}r = \frac{4\mu}{\pi}\int_a^t \frac{u_\theta r^2}{(t^2 - r^2)^{1/2}}\,\mathrm{d}r. \tag{3.5}$$

If we differentiate (3.4) with respect to t, we obtain for $t < a$

$$\int_t^a \frac{\tau_{z\theta}}{(r^2 - t^2)^{1/2}}\,\mathrm{d}r = \int_a^\infty \frac{\tau(r)}{(r^2 - t^2)^{1/2}}\,\mathrm{d}r. \tag{3.6}$$

Equation (3.6), which holds for $t < a$, is an Abel integral equation for $\tau_{z\theta}$ and the solution is straightforward. We have for $r < a$, $z = 0$

$$\tau_{z\theta} = \frac{2r}{\pi(a^2 - r^2)^{1/2}} \int_a^\infty \frac{(s^2 - a^2)^{1/2}}{s^2 - r^2} \tau(s)\,ds. \tag{3.7}$$

The torque M_t transmitted through the neck is found to be

$$M_t = 2\pi \int_0^a \tau_{z\theta} r^2\,dr = 4 \int_a^\infty \tau(r)\left[r \sin^{-1}\left(\frac{a}{r}\right) - \frac{a}{r}(r^2 - a^2)^{1/2}\right] r\,dr. \tag{3.8}$$

The stress intensity factor k_3 is

$$k_3 = \lim_{r \to a} [2(a - r)]^{1/2} \tau_{z\theta} = \frac{2a^{1/2}}{\pi} \int_a^\infty \frac{\tau(s)}{(s^2 - a^2)^{1/2}}\,ds \tag{3.9}$$

which agrees with the known result[8]. Substituting (3.8) into (3.5) and solving the resulting equation for u_θ, we obtain for $r \ge a$, $z = 0$

$$u_\theta = \frac{-2}{\pi\mu r} \int_a^r \frac{t^2\,dt}{(r^2 - t^2)^{1/2}} \int_t^\infty \frac{\tau(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda. \tag{3.10}$$

For a flat toroidal crack with outer crack radius b, we suppose that $\tau_{z\theta}$ is prescribed for $a \le r \le b$ and in addition to u_θ vanishing for $r \le a$ on $z = 0$ we require

$$u_\theta = 0, \quad \text{for } r \ge b,\ z = 0. \tag{3.11}$$

We denote the unknown value of $\tau_{z\theta}$ on $z = 0$ outside $r = b$ by $f(r)$. From (3.10) and (3.11) it follows that $f(r)$ must be such that

$$\int_b^r \frac{t^2\,dt}{(r^2 - t^2)^{1/2}} \int_t^\infty \frac{f(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda + \int_a^b \frac{t^2\,dt}{(r^2 - t^2)^{1/2}} \int_b^\infty \frac{f(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda = \int_a^b \frac{t^2\,dt}{(r^2 - t^2)^{1/2}} \int_t^b \frac{\tau(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda. \tag{3.12}$$

Following Cooke[9], we set

$$F(s) = \int_s^\infty \frac{f(\lambda)}{(\lambda^2 - s^2)^{1/2}}\,d\lambda, \tag{3.13}$$

and then

$$f(\lambda) = -\frac{2}{\pi} \frac{d}{d\lambda} \int_\lambda^\infty \frac{sF(s)}{(s^2 - \lambda^2)^{1/2}}\,ds. \tag{3.14}$$

On using (3.13) and (3.14) we can show that eqn (3.12) can be reduced to a Fredholm integral equation of the second kind

$$t(t^2 - b^2)^{1/2} F(t) = \frac{2}{\pi} \int_a^b \frac{\lambda^2 (b^2 - \lambda^2)^{1/2}}{t^2 - \lambda^2}\,d\lambda \int_\lambda^b \frac{\tau(s)}{(s^2 - \lambda^2)^{1/2}}\,ds - \frac{4}{\pi^2} \int_b^\infty K(t, s) F(s)\,ds \quad \text{for } t > b, \tag{3.15}$$

where

$$K(t, s) = \frac{s}{(s^2 - b^2)^{1/2}} \left\{-b + a + \frac{1}{2(t^2 - s^2)}\left[t(t^2 - b^2) \ln\frac{(t + b)(t - a)}{(t - b)(t + a)} - s(s^2 - b^2) \ln\frac{(s + b)(s - a)}{(s - b)(s + a)}\right]\right\}. \tag{3.16}$$

Numerical solution of the integral equation (3.15) is not straightforward because of the singularity at $t = b$. We first use the result for a penny-shaped crack[1]

$$f(\lambda) = \tau\left[\frac{(\lambda^2 - b^2)^{1/2}}{\lambda} + \frac{b^2}{2\lambda(\lambda^2 - b^2)^{1/2}} - 1\right], \quad \lambda > b, \tag{3.17}$$

where τ is the constant value of $\tau(r)$ for $r < b$. Substituting (3.17) into (3.13), we have for this case

$$F(s) = -\frac{\tau}{4}\left(\frac{b}{s}\ln\frac{s+b}{s-b} + 2\ln\frac{s^2 - b^2}{s^2}\right), \tag{3.18}$$

which suggests that we set

$$F(s) = \left[\frac{b}{s}\ln\frac{s+b}{s-b} + 2\ln\frac{s^2 - b^2}{s^2}\right]G(s), \tag{3.19}$$

where $G(s)$ is a bounded function for $s \geq b$. Substituting (3.19) into (3.14), we have as $\lambda \to b$

$$f(\lambda) = -\frac{2bG(b)}{(\lambda^2 - b^2)^{1/2}} + O(1). \tag{3.20}$$

From (3.20) the stress intensity factor k_b at the outer tip is found to be

$$k_b = \lim_{r\to b}[2(r-b)]^{1/2}f(r) = -2b^{1/2}G(b). \tag{3.21}$$

The stress $\tau_{z\theta}$ for $r < a$ is calculated from (3.7) and (3.14) so that as $r \to a$

$$k_a = \frac{2(\beta b)^{1/2}}{\pi}\left\{\int_a^b \frac{\tau(s)}{(s^2 - a^2)^{1/2}}\,ds - \frac{2b}{\pi}(1-\beta^2)^{1/2}\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2 - b^2)^{1/2}(\lambda^2 - a^2)}\,d\lambda\right\}, \tag{3.22}$$

where $\beta = a/b$. The displacement u_θ on the crack surface $a \leq r \leq b$, $z = 0$ is obtained from (3.10) and (3.14)

$$u_\theta = -\frac{2}{\pi\mu r}\left\{\int_a^r \frac{t^2\,dt}{(r^2 - t^2)^{1/2}}\int_t^b \frac{\tau(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda - \frac{2}{\pi}\int_a^r \frac{t^2(b^2 - t^2)^{1/2}}{(r^2 - t^2)^{1/2}}\,dt\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2 - b^2)^{1/2}(\lambda^2 - t^2)}\,d\lambda\right\}. \tag{3.23}$$

As indicated by (3.19), the solution $F(t)$ of the integral equation (3.15) has a logarithmic singularity at $t = b$. One approach is to determine the stress intensity factor k_b by numerical extrapolation, but the accuracy is not high for some values of a/b. It turns out that with this approach the value of k_b in (3.21) is very sensitive to the form assumed for $F(t)$, such as (3.19). To avoid this difficulty we apply the M-integral in (2.2) to the field by taking A as the union of a closed surface consisting of the crack surfaces and two toroids with vanishing radii surrounding the crack tips, and the surface of the sphere at infinity. We obtain

$$b^2J_b - a^2J_a = -\int_a^b \left\{3r\tau(r) + 2r^2\frac{\partial\tau(r)}{\partial r}\right\}u_\theta\,dr \tag{3.24}$$

where J_a, J_b denote the J-integrals at $r = a$ and $r = b$. Substituting (3.23) into (3.24), we have

$$b^2J_b - a^2J_a = \frac{2}{\pi\mu}\int_a^b\left[3\tau(r) + 2r\frac{\partial\tau(r)}{\partial r}\right]dr\int_a^r \frac{t^2\,dt}{(r^2 - t^2)^{1/2}}\int_t^b \frac{\tau(\lambda)}{(\lambda^2 - t^2)^{1/2}}\,d\lambda$$
$$- \frac{4}{\pi^2\mu}\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2 - b^2)^{1/2}}\,d\lambda\int_a^b\left[3\tau(r) + 2r\frac{\partial\tau(r)}{\partial r}\right]dr\int_a^r \frac{t^2(b^2 - t^2)^{1/2}}{(r^2 - t^2)^{1/2}(\lambda^2 - t^2)}\,dt. \tag{3.25}$$

Since k_1 and k_2 vanish, we also have

$$J_a = \frac{\pi}{2\mu} k_a^2, \quad J_b = \frac{\pi}{2\mu} k_b^2 \tag{3.26}$$

and the eqn (3.25) provides the relation between k_a and k_b. With (3.22) we can now determine more accurate estimates for k_b.

For a numerical example we assume

$$\tau = qr/b, \tag{3.27}$$

where q is a constant, which corresponds to torsion of a circular cylinder with an infinitesimal central crack perpendicular to the axis. The numerical results are shown in Fig. 1. As $a/b \to 0$, k_b approaches the value

$$k_b = \frac{4qb^{1/2}}{3\pi}$$

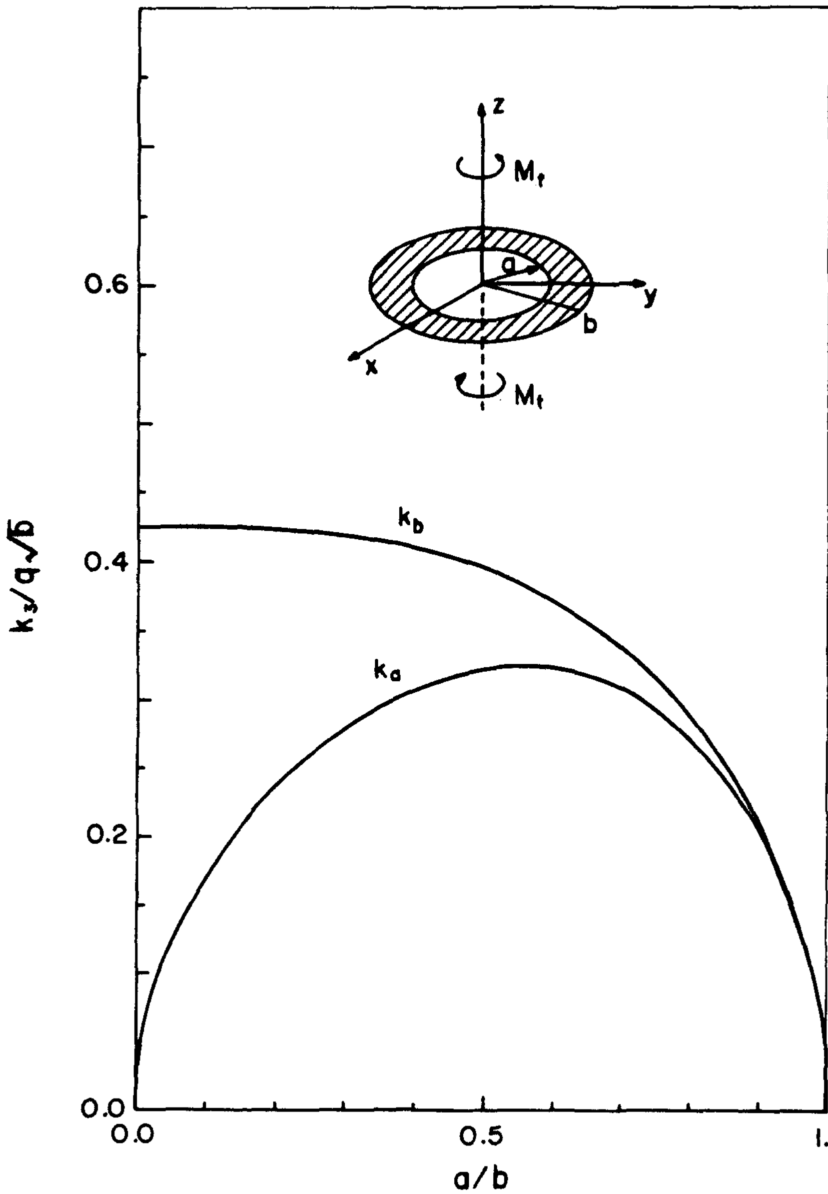

Fig. 1. Stress intensity factor for toroidal crack under torsion

which is the solution for a penny-shaped crack[8], and as $a/b \to 1$, k_a and k_b tend to

$$k_a = k_b = q\left(\frac{b-a}{2}\right)^{1/2}$$

which is the solution for antiplane shear. It is seen that k_b is always greater than k_a, so it is to be expected that if growth of the crack occurs it will be at the outer edge.

We remark that the M-integral has been used by Freund[10] to find stress intensity factors for plane cracks.

4. APPLICATION OF PRESSURE ON A FLAT TOROIDAL CRACK

For a flat toroidal crack under pressure we use the solution for indentation of a half space by a circular punch. Apart from a multiplicative constant, surface values are given by

$$\begin{aligned} \tau'_{33} &= \frac{2\mu}{\pi(1-\nu)(t^2-r^2)^{1/2}}, \quad u'_3 = 1 \quad \text{for } r < t, \\ \tau'_{33} &= 0, \quad u'_3 = \frac{2}{\pi}\sin^{-1}\left(\frac{t}{r}\right) \quad \text{for } t < r \end{aligned} \tag{4.1}$$

on $z = 0$, where t is the punch radius (see, e.g. [2]). By a similar method to that of the previous section we can use the reciprocal theorem (2.1) to obtain the known results for an external crack [11]

$$\begin{aligned} \tau_{33} &= \frac{2}{\pi(a^2-r^2)^{1/2}}\int_a^\infty \frac{(s^2-a^2)^{1/2}}{s^2-r^2} sp(s)\,\mathrm{d}s \quad \text{for } r < a,\ z = 0, \\ u_3 &= -\frac{2(1-\nu)}{\pi\mu}\int_a^r \frac{\mathrm{d}t}{(r^2-t^2)^{1/2}}\int_t^\infty \frac{sp(s)}{(s^2-t^2)^{1/2}}\,\mathrm{d}s \quad \text{for } r > a,\ z = 0, \end{aligned} \tag{4.2}$$

where $p(r)$ denotes the prescribed pressure on the crack surface.

For an annular crack we set

$$u_3 = 0, \quad p(r) = -f(r) \quad \text{for } r > b,\ z = 0, \tag{4.3}$$

where $f(r)$ denotes the unknown values of τ_{33} on $z = 0$. Again following the procedure described in Section 3, we obtain an integral equation

$$F(t) = \frac{2t}{\pi(t^2-b^2)^{1/2}}\int_a^b \frac{(b^2-\lambda^2)^{1/2}}{t^2-\lambda^2}\,\mathrm{d}\lambda \int_\lambda^b \frac{sp(s)}{(s^2-\lambda^2)^{1/2}}\,\mathrm{d}s - \frac{4}{\pi^2}\int_b^\infty K(t,s)F(s)\,\mathrm{d}s, \tag{4.4}$$

where

$$K(t,s) = \frac{1}{2(t^2-s^2)}\left\{\frac{s(t^2-b^2)^{1/2}}{(s^2-b^2)^{1/2}}\ln\frac{(t+b)(t-a)}{(t-b)(t+a)} - \frac{t(s^2-b^2)^{1/2}}{(t^2-b^2)^{1/2}}\ln\frac{(s+b)(s-a)}{(s-b)(s+a)}\right\}, \tag{4.5}$$

and

$$F(s) = \int_s^\infty \frac{\lambda f(\lambda)}{(\lambda^2-s^2)^{1/2}}\,\mathrm{d}\lambda \tag{4.6}$$

so that

$$f(\lambda) = -\frac{2}{\pi\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_\lambda^\infty \frac{sF(s)}{(s^2-\lambda^2)^{1/2}}\,\mathrm{d}s. \tag{4.7}$$

We assume

$$F(t) = b\left[\frac{t}{b}\ln\frac{t+b}{t-b} - 2\right]G(t) \tag{4.8}$$

and then

$$k_b = \lim_{r\to b} [2(r-b)]^{1/2}\tau_{33} = 2b^{1/2}G(b) \tag{4.9}$$

and

$$k_a = \frac{2b^{1/2}}{\pi\beta^{1/2}}\left\{\int_a^b \frac{sp(s)}{b(s^2-a^2)^{1/2}}\,\mathrm{d}s - \frac{2}{\pi}(1-\beta^2)^{1/2}\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2-b^2)^{1/2}(\lambda^2-a^2)}\,\mathrm{d}\lambda\right\}. \tag{4.10}$$

The displacement on the crack surface is found to be

$$u_3 = -\frac{2(1-\nu)}{\pi\mu}\left\{\int_a^r \frac{\mathrm{d}t}{(r^2-t^2)^{1/2}}\int_t^b \frac{sp(s)}{(s^2-t^2)^{1/2}}\,\mathrm{d}s - \frac{2}{\pi}\int_a^r \frac{(b^2-t^2)^{1/2}}{(r^2-t^2)^{1/2}}\,\mathrm{d}t\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2-b^2)^{1/2}(\lambda^2-t^2)}\,\mathrm{d}\lambda\right\}. \tag{4.11}$$

We also have for this case, using the M-integral of (2.2),

$$\begin{aligned} b^2J_b - a^2J_a = {} & \frac{2(1-\nu)}{\pi\mu}\int_a^b \left[3rp(r) + 2r^2\frac{\partial p(r)}{\partial r}\right]\mathrm{d}r\int_a^r \frac{\mathrm{d}t}{(r^2-t^2)^{1/2}}\int_t^b \frac{sp(s)}{(s^2-t^2)^{1/2}}\,\mathrm{d}s \\ & -\frac{4(1-\nu)}{\pi^2\mu}\int_b^\infty \frac{\lambda F(\lambda)}{(\lambda^2-b^2)^{1/2}}\,\mathrm{d}\lambda\int_a^b\left[3rp(r)\right. \\ & \left. + 2r^2\frac{\partial p(r)}{\partial r}\right]\mathrm{d}r\int_a^r \frac{(b^2-t^2)^{1/2}}{(r^2-t^2)^{1/2}(\lambda^2-t^2)}\,\mathrm{d}t. \end{aligned} \tag{4.12}$$

Since $k_2 = k_3 = 0$, the eqn (4.12) provides the relation between k_a and k_b.

Numerical results for uniform pressure p are given in Table 1 which compares them with values given by Smetanin[12] and values provided by expressions given by Moss and Kobayashi[13]. It is seen that the present calculation agrees well with the results of Smetanin while the agreement with the results of Moss and Kobayashi is poor. Figure 2 shows the stress intensity factors versus a/b. The maximum mismatch with the asymptotic formulae given by Smetanin is less than 1%. As we might expect, k_b approaches the value of

$$k_b = \frac{2}{\pi}pb^{1/2}$$

for a penny-shaped crack and $k_a \to \infty$ as $a/b \to 0$, whereas

$$k_a = k_b = p\left(\frac{b-a}{2}\right)^{1/2}$$

as $a/b \to 1$. Since $k_a > k_b$, growth of the crack under mode I loading would tend to occur at the inner edge suggesting that an annular crack will develop into a penny-shaped crack. This is in contrast to the result of the previous section which indicated growth of the crack at the outer edge under shear induced by torsion.

Table 1. Stress intensity factors for uniform pressure in a flat toroidal crack

a/b	$k_a/pb^{1/2}$			$k_b/pb^{1/2}$		
	Present	From [12]	From [13]	Present	From [12]	From [13]
0.06948	1.575	1.574	0.863	0.618	0.618	0.618
0.36788	0.693	0.697	0.742	0.525	0.526	0.543

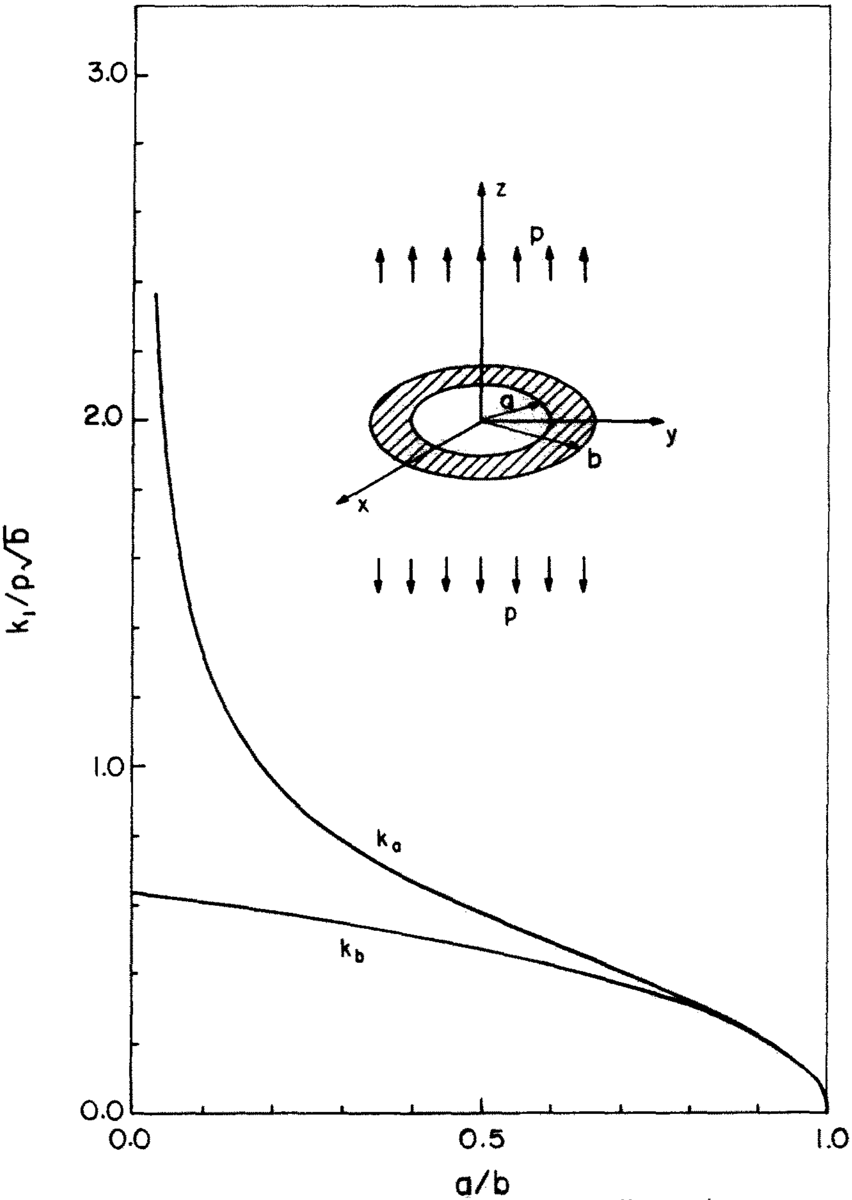

Fig. 2. Stress intensity factor for toroidal crack under uniform tension

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